Algebarski izrazi

$(a+b)^2 = a^2 + 2ab + b^2$, $\quad (a+b)^2 = (a+b)(a+b)$

$(a-b)^2 = a^2 - 2ab + b^2$, $\quad (a-b)^2 = (a-b)(a+b)$

$a^2 + b^2$

$a^2 - b^2 = (a-b)(a+b)$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

1. Uprostiti izraz: $\left(\frac{3}{a-1} + \frac{3a^2+3a+3}{1-a^2} : \frac{a^4-a}{a^3+1}\right) \cdot \frac{a-a^2}{3}$.

Rj.

$$\left(\frac{3}{a-1} + \frac{3a^2+3a+3}{1-a^2} : \frac{a^4-a}{a^3+1}\right) \cdot \frac{a-a^2}{3} =$$

$$= \left(\frac{3}{a-1} + \frac{3(a^2+a+1)}{-(a^2-1)} \cdot \frac{a^3+1}{a(a^3-1)}\right) \cdot \frac{-(a^2-a)}{3} =$$

$$= \left(\frac{3}{a-1} + \frac{3\cancel{(a^2+a+1)}}{-(a-1)\cancel{(a+1)}} \cdot \frac{\cancel{(a+1)}(a^2-a+1)}{a(a-1)\cancel{(a^2+a+1)}}\right) \cdot \frac{(-a)(a-1)}{3}$$

$$= \left(\frac{3}{a-1} + \frac{3(a^2-a+1)}{(-a)(a-1)^2}\right) \cdot \frac{(-a)(a-1)}{3} =$$

$$= \frac{3\cdot(-a)(a-1) + 3(a^2-a+1)}{\cancel{(-a)}(a-1)^{\cancel{2}\,1}} \cdot \frac{\cancel{(-a)}\cancel{(a-1)}}{3} = \frac{3(-a^2+a+a^2-a+1)}{3(a-1)} = \frac{1}{a-1}$$

2. Uprostiti izraz: $\left[\frac{1}{(b-a)^3} : \frac{1}{(a-b)^2} - \frac{1}{a+b}\right] : \frac{2a^2}{a^2-b^2} + 1 : (a^2+a)$;

Rj.

$$\left[\frac{1}{(b-a)^3} : \frac{1}{(a-b)^2} - \frac{1}{a+b}\right] : \frac{2a^2}{a^2-b^2} + 1 : (a^2+a) =$$

$$= \left[\frac{1}{((-1)(a-b))^3} \cdot \frac{(a-b)^2}{1} - \frac{1}{a+b}\right] \cdot \frac{a^2-b^2}{2a^2} + \frac{1}{a^2+a} =$$

$$= \left[\frac{(a-b)^2}{(-1)(a-b)^3} - \frac{1}{a+b}\right] \cdot \frac{a^2-b^2}{2a^2} + \frac{1}{a(a+1)} =$$

$$= \left[\frac{(-1)}{a-b} + \frac{(-1)}{a+b}\right] \cdot \frac{(a-b)(a+b)}{2a^2} + \frac{1}{a(a+1)} = \frac{-a-b-a+b}{(a-b)(a+b)} \cdot \frac{(a-b)(a+b)}{2a^2} + \frac{1}{a(a+1)} =$$

$$= \frac{-2a}{2a^2} + \frac{1}{a(a+1)} = \frac{(-1)}{a} + \frac{1}{a(a+1)} = \frac{-a-1+1}{a(a+1)} = \frac{-1}{a+1}$$

3. Uprostiti izraz: $\dfrac{(\sqrt{a}-\sqrt{b})^3 + 2\sqrt{a^3} + b\sqrt{b}}{a\sqrt{a} + b\sqrt{b}} + \dfrac{3\sqrt{ab}-3b}{a-b}$

Rj.

$$\frac{(\sqrt{a}-\sqrt{b})^3 + 2\sqrt{a^3} + b\sqrt{b}}{a\sqrt{a} + b\sqrt{b}} + \frac{3\sqrt{ab}-3b}{a-b} =$$

$$= \frac{\sqrt{a^3} - 3\sqrt{a^2b} + 3\sqrt{ab^2} - \sqrt{b^3} + 2\sqrt{a^3} + \sqrt{b^3}}{\sqrt{a^3}+\sqrt{b^3}} + \frac{3\sqrt{ab}-3\sqrt{b^2}}{\sqrt{a^2}-\sqrt{b^2}} =$$

$$= \frac{3\sqrt{a^3} - 3\sqrt{a^2b} + 3\sqrt{ab^2}}{\sqrt{a^3}+\sqrt{b^3}} + \frac{3\sqrt{b}\cancel{(\sqrt{a}-\sqrt{b})}}{\cancel{(\sqrt{a}-\sqrt{b})}(\sqrt{a}+\sqrt{b})} =$$

$$= \frac{3\sqrt{a}\cancel{(\sqrt{a^2}-\sqrt{ab}+\sqrt{b^2})}}{(\sqrt{a}+\sqrt{b})\cancel{(\sqrt{a^2}-\sqrt{ab}+\sqrt{b^2})}} + \frac{3\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{3\sqrt{a}+3\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{3(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = 3$$

4. Uprostiti izraz: $\left(\dfrac{m^{\frac{2}{3}}}{m^{\frac{2}{3}} - 2m^{-\frac{1}{3}}} - \dfrac{m^{\frac{4}{3}}}{m^{\frac{4}{3}} - m^{\frac{1}{3}}}\right)(m-3+2m^{-1}) - \left(\dfrac{2m-3}{m+5}\right)^0$.

$$\left(\frac{m^{\frac{2}{3}}}{m^{\frac{2}{3}} - 2m^{-\frac{1}{3}}} - \frac{m^{\frac{4}{3}}}{m^{\frac{4}{3}} - m^{\frac{1}{3}}}\right)(m-3+2m^{-1}) - \left(\frac{2m-3}{m+5}\right)^0 =$$

$$= \left(\frac{\sqrt[3]{m^2}}{\sqrt[3]{m^2} - \frac{2}{\sqrt[3]{m}}} - \frac{\sqrt[3]{m^4}}{\sqrt[3]{m^4} - \sqrt[3]{m}}\right)\left(m-3+\frac{2}{m}\right) - 1 =$$

$$= \left(\frac{\sqrt[3]{m^2}}{\frac{\sqrt[3]{m^3}-2}{\sqrt[3]{m}}} - \frac{\sqrt[3]{m^3}\cdot\cancel{\sqrt[3]{m}}}{\cancel{\sqrt[3]{m}}(\sqrt[3]{m^3}-1)}\right)\frac{m^2-3m+2}{m} - 1 =$$

$$= \left(\frac{\sqrt[3]{m^3}}{\sqrt[3]{m^3}-2} - \frac{\sqrt[3]{m^3}}{\sqrt[3]{m^3}-1}\right) \cdot \frac{m^2-m-2m+2}{m} - 1 = \left(\frac{m}{m-2} - \frac{m}{m-1}\right) \cdot \frac{m(m-1)-2(m-1)}{m} - 1 =$$

$$= \frac{m(m-1)-m(m-2)}{\cancel{(m-2)}\cancel{(m-1)}} \cdot \frac{\cancel{(m-2)}\cancel{(m-1)}}{m} - 1 = \frac{\cancel{m}(m-1-m+2)}{\cancel{m}} - 1 = 1-1=0$$

5. Uprostiti izraz: $\frac{\sqrt{a}-\sqrt{x}}{\sqrt[4]{a}-\sqrt[4]{x}} - \left(\frac{a+\sqrt[4]{ax^3}}{\sqrt{a}+\sqrt[4]{ax}} - \sqrt[4]{ax}\right) : \left(\sqrt[4]{a}-\sqrt[4]{x}\right)$.

6. Uprostiti izraz: $\left[\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)^{-1}\left(a^{\frac{3}{2}}-b^{\frac{3}{2}}\right) - \frac{1}{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}\right] : \sqrt[3]{ab\sqrt{ab}} + \frac{1}{1+\left[a\left(1-a^2\right)^{-\frac{1}{2}}\right]^2}$.

7. Uprastiti izraz: $\left(\frac{\sqrt[4]{a^3}-1}{\sqrt[4]{a}-1} + \sqrt[4]{a}\right)^{\frac{1}{2}} \cdot \left(\frac{\sqrt[4]{a^3}+1}{\sqrt[4]{a}+1} - \sqrt{a}\right) \cdot \left(a-\sqrt{a^3}\right)^{-1}$.

Rješenja: 5. $2\sqrt[4]{x}$ 6. $-a^2$ 7. $\frac{1}{a}$

Kvadratne jednačine i kvadratna f-ja

Jednačina oblika $ax^2+bx+c=0$ ($a,b,c \in R$, $a \neq 0$) zove se kvadratna jednačina.

F-ja $f: R \to R$ gdje je $f(x)=ax^2+bx+c$ (drugačije napisano $y=ax^2+bx+c$) $a,b,c \in R$, $a\neq 0$ zove se kvadratna f-ja ili polinom drugog stepena.

1. Riješiti kvadratne jednačine:

a) $(2x-3)^2=15$ b) $4x^2+9=0$ c) $5x^2-7x=0$.

Rj. a) $(2x-3)^2=15$

$2x-3=\pm\sqrt{15}$

$2x=\pm\sqrt{15}+3$

$x=\pm\frac{\sqrt{15}}{2}+\frac{3}{2}$

$x_1=-\frac{\sqrt{15}}{2}+\frac{3}{2}$

$x_2=\frac{\sqrt{15}}{2}+\frac{3}{2}$

b) $4x^2+9=0$

$4x^2=-9$

$x^2=-\frac{9}{4}$

$x=\pm\sqrt{-\frac{9}{4}}$

$x=\pm\sqrt{\frac{9}{4}i^2}$

$x_1=-\frac{3}{2}i$

$x_2=\frac{3}{2}i$

c) $5x^2-7x=0$

$(5x-7)x=0$

$5x-7=0$ ili $x=0$

$5x=7$

$x=\frac{7}{5}$

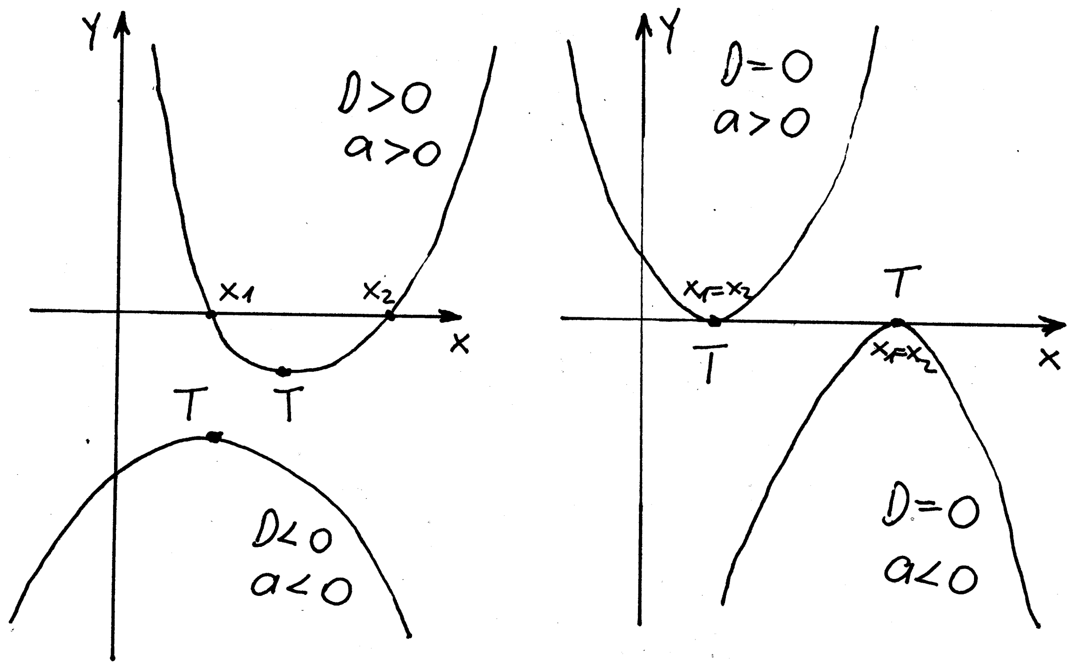
Rješenje kvadratne jednačine je $x=\frac{7}{5}$ ili $x=0$.

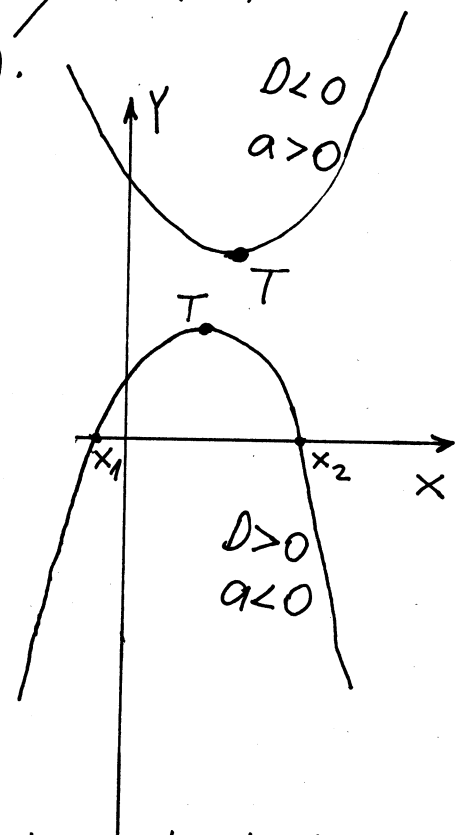
$D = b^2 - 4ac$, D diskriminanta

Grafik kvadratne f-je $f(x) = ax^2 + bx + c$ ima oblik parabole koja ima nule u tačkama $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$.

Ako je $a > 0$ minimum f-je je u tački $T\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$.

Ako je $a < 0$ kvadratna f-ja ima maksimum u tački $T\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$ (u istoj tački).





Primjetimo da:

$$D = b^2 - 4ac = \begin{cases} >0, & x_1 \neq x_2 \text{ realni različiti brojevi} \\ =0, & x_1 = x_2 \text{ realni brojevi} \\ <0, & x_1, x_2 \text{ konjugovano kompleksni brojevi} \end{cases}$$

2. Grafički predstaviti i naći ekstrem f-je $y = x^2 - 6x + 8$.

Rj. Tražimo nule f-je (u kojim tačkama f-ja siječe x-osu)

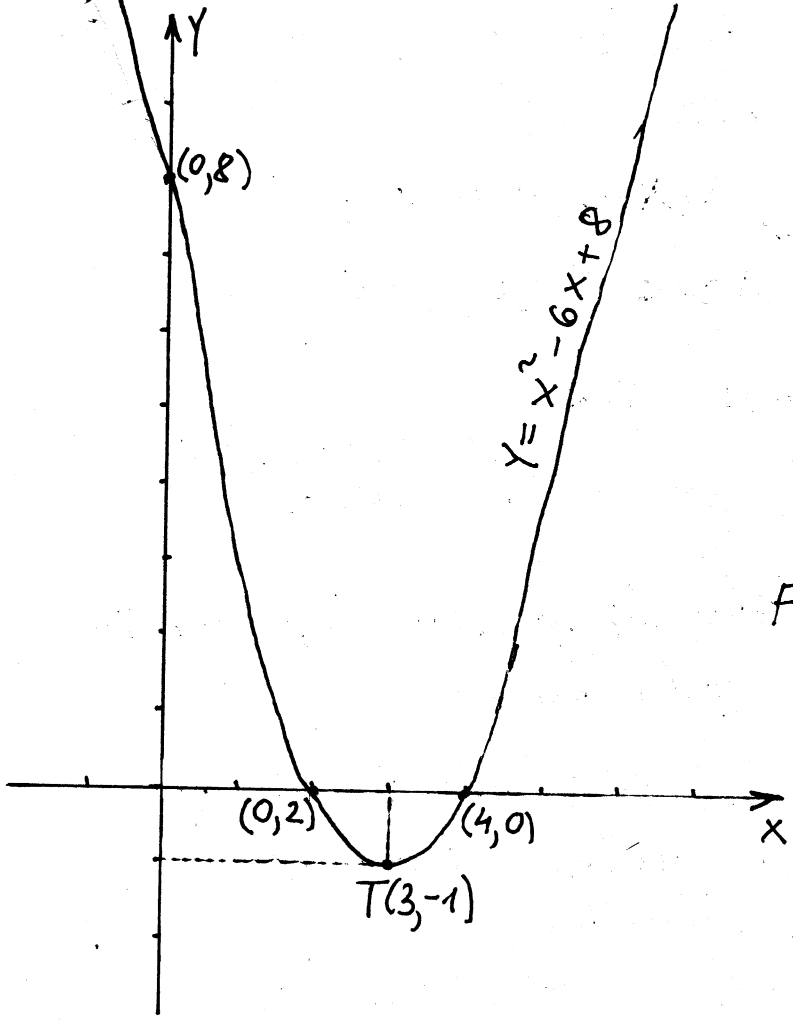
$x^2 - 6x + 8 = 0$

$D = 36 - 32 = 4$

$x_{1,2} = \dfrac{6 \pm 2}{2}$

$x_1 = \dfrac{4}{2} = 2 \quad x_2 = \dfrac{8}{2} = 4$

Nule f-je su $x_1 = 2$; $x_2 = 4$



Tražimo presjek sa Y-osom.

$f(x)=x^2-6x+8$

$f(0)=8$

(0,8) je tačka presjeka grafa f-je sa Y-osom.

Tražimo ekstreme f-je

$a=1>0 \Rightarrow$ f-ja je oblika $\cup$

F-ja ima minimum u tački

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a}=-\frac{(-6)}{2}=3$

$-\frac{D}{4a}=-\frac{4}{4}=-1 \qquad T(3,-1)$

Jednačinu $ax^2+bx+c=0$ možemo rastaviti na faktore pomoću formule $a(x-x_1)(x-x_2)=0$. $\left(x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$.

3. Sljedeće jednačine rastaviti na faktore:

a) $3x^2+5x-8=0$ b) $2x^2+13x-7=0$ c) $6x^2-x-2=0$.

Rj. a) $3x^2+5x-8=0$

$D=25+96=121$

$x_{1,2}=\frac{-5\pm 11}{6}$

$x_1=\frac{-16}{6}=-\frac{8}{3}$

$x_2=\frac{6}{6}=1$

$3\left(x+\frac{8}{3}\right)(x-1)=0$

Jednačina rastavljena na faktore je

$(3x+8)(x-1)=0$

b) $2x^2+13x-7=0$

$D=169+56$

$x_{1,2}=\frac{-13\pm 15}{4}$

$x_1=\frac{-28}{4}=-7$

$x_2=\frac{2}{4}=\frac{1}{2}$

$2(x+7)\left(x-\frac{1}{2}\right)=0$

Jednačina rastavljena na faktore je

$(x+7)(2x-1)=0$

c) $6x^2-x-2=0$

$D=1+48=49$

$x_{1,2}=\frac{1\pm 7}{12}$

$x_1=\frac{-6}{12}=-\frac{1}{2}$

$x_2=\frac{8}{12}=\frac{2}{3}$

$6\left(x+\frac{1}{2}\right)\left(x-\frac{2}{3}\right)=$

$2\cdot 3\left(x+\frac{1}{2}\right)\left(x-\frac{2}{3}\right)=$

$=2\left(x+\frac{1}{2}\right)\cdot 3\left(x-\frac{2}{3}\right)=$

$=(2x+1)(3x-2)$

4. Za koju vrijednost parametra λ jednačina $\lambda^2(x-1)=4\lambda(x-2)+16$ ima više od jednog rješenja.

Rj. $\lambda^2(x-1)=4\lambda(x-2)+16$

$\underline{\lambda^2 x}-\lambda^2=\underline{4\lambda x}-8\lambda+16$

$\lambda^2 x-4\lambda x=\lambda^2-8\lambda+16$

$\lambda(\lambda-4)x=(\lambda-4)^2$

$\lambda=0$: $0\cdot x=16$ nema rješenja

$\lambda=4$: $0\cdot x=0$ ∞ mnogo rješenja

Za $\lambda=4$ jednačina ima ∞ mnogo rješenja.

5. Odrediti parametar λ tako da rješenja jednačine $8(x^2-1)=(\lambda-2)x-\lambda$ budu jednaka.

Rj. $8(x^2-1)=(\lambda-2)x-\lambda$

$8x^2-8-(\lambda-2)x+\lambda=0$

$8x^2+(-\lambda+2)x+\lambda-8=0$

$D=(-\lambda+2)^2-4\cdot 8\cdot(\lambda-8)$

$=\lambda^2-4\lambda+4-32\lambda+256$

$=\lambda^2-36\lambda+260$

Za $D=0$ rješenja svake kvadratne jednačine su jednaka.

$\lambda^2-36\lambda+260=0$

$D=1296-1040=256$

$\lambda_{1,2}=\frac{36\pm 16}{2}$

$\lambda_1=\frac{20}{2}=10$ $\lambda_2=\frac{52}{2}=26$

Rješenja jednačine će biti jednaka za $\lambda=10$ ili za $\lambda=26$.

6. Grafički predstaviti i naći ekstrem f-je

a) $y=-\frac{1}{2}x^2+x+1\frac{1}{2}$

b) $y=2x^2+9x-5$

7. Rastaviti na faktore

a) $6x^2+5bx+b^2=0$

b) $8x^2+2px-3p^2=0$

8. Za koje vrijednosti parametra λ su rješenja jednačine $(1-\lambda)x^2-2(1+\lambda)x+3(1+\lambda)=0$ realna i različita.

Rješenja:

6. a) $T(1,2)$ b) $T(-2\frac{1}{4},-15\frac{1}{8})$

7. a) $(2x+b)(3x+b)=0$

b) $(4x+3p)(2x-p)=0$

8. $\lambda\in(-\infty,-1)\cup(\frac{1}{2},+\infty)$

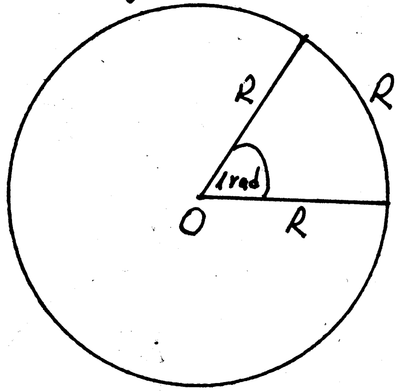
Trigonometrija

Najpoznatije jedinice za mjerenje veličine ugla su radijan i stepen.

2π rad $= 360°$ $\quad \frac{\pi}{2}$ rad $= 90°$ $\quad \frac{\pi}{3}$ rad $= 60°$

π rad $= 180°$ $\quad \frac{\pi}{4}$ rad $= 45°$ $\quad \frac{\pi}{6}$ rad $= 30°$

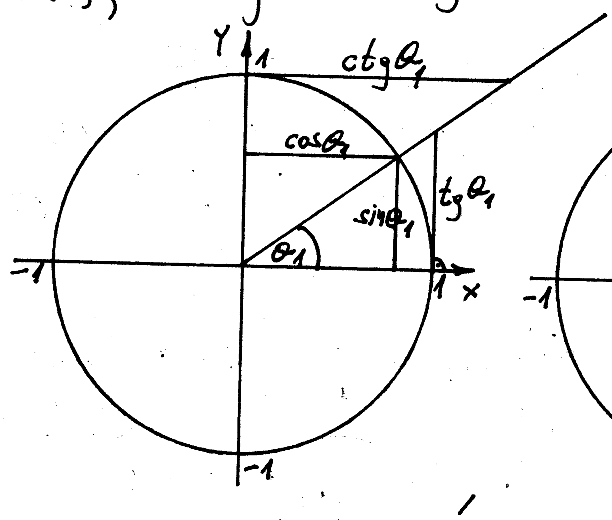
Stepen je devedeseti dio pravog ugla.

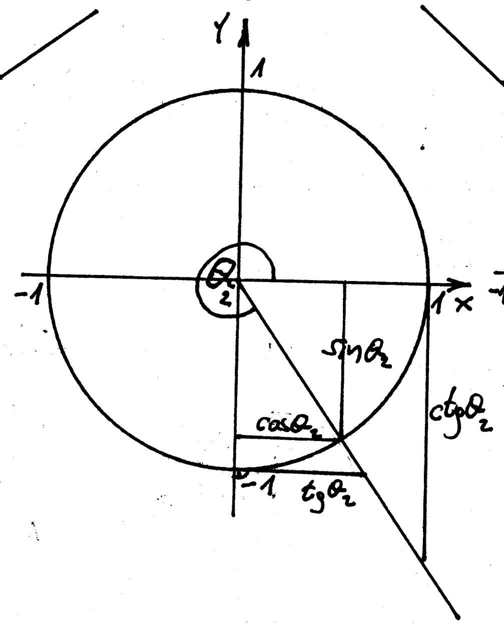


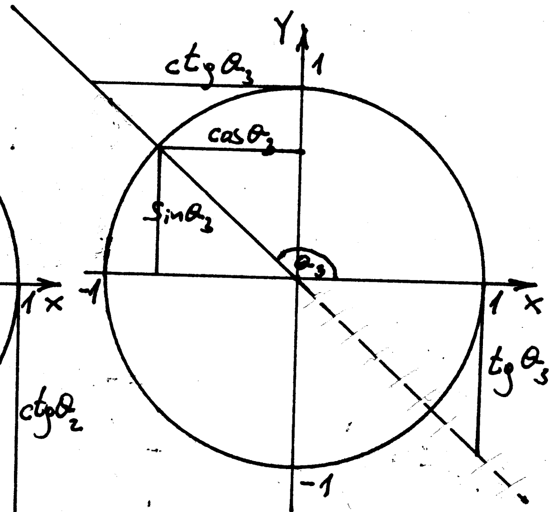
Radijan je veličina centralnog ugla nad lukom (kružnice) čija je dužina jednaka poluprečniku (slika).

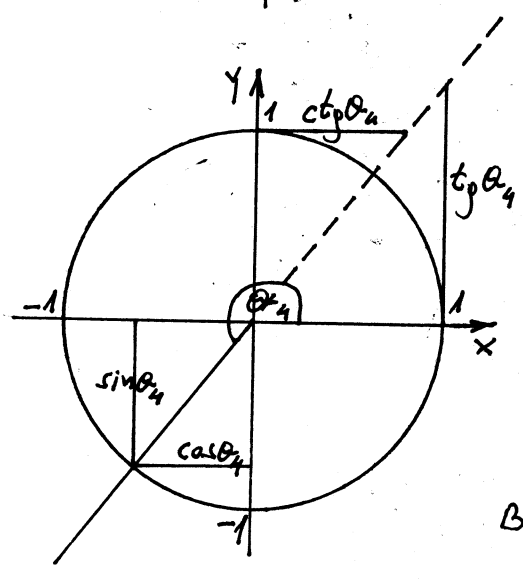
$$1\text{ rad} = \frac{360°}{2\pi} \qquad 1\text{ rad} \approx 57°17'44''$$

Krug sa centrom u koordinatnom početku poluprečnika 1 (jedan) nam pomaže da definišemo sinus (sin), kosinus (cos), tangens (tg) i kotangens (ctg) proizvoljnog ugla.

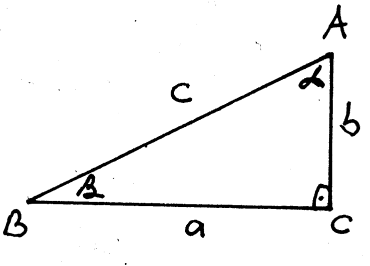








Ako je dat pravougli trougao:

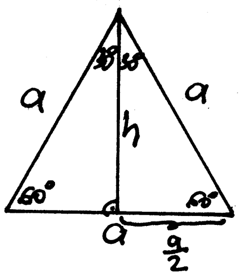


$$\sin\alpha = \frac{a}{c}, \quad \sin\beta = \frac{b}{c},$$

$$\cos\alpha = \frac{b}{c}, \quad \cos\beta = \frac{a}{c}$$

$$tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad ctg\,\alpha = \frac{\cos\alpha}{\sin\alpha}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$



$\sin 60^\circ = \frac{h}{a}$

$h^2 = a^2 - \frac{a^2}{4}$

$h = \frac{a\sqrt{3}}{2}$

$\sin 60^\circ = \frac{\frac{a\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{2}$

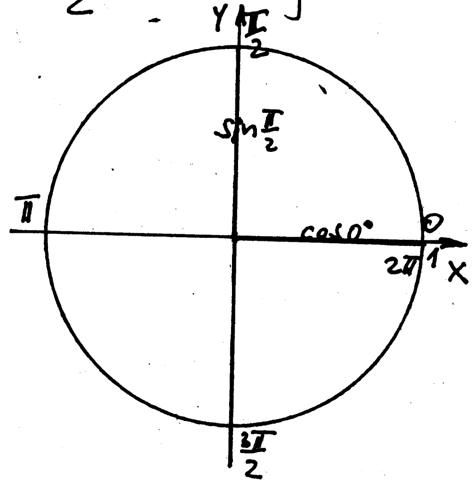
$\cos 60^\circ = \frac{\frac{a}{2}}{a} = \frac{1}{2}$

$\sin 30^\circ = \frac{\frac{a}{2}}{a} = \frac{1}{2}$

$\cos 30^\circ = \frac{h}{a} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$

α	30°	60°	45°
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
$tg\,\alpha$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	1
$ctg\,\alpha$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1

1. Izračunati: a) $\cos 0^\circ$ b) $\sin\frac{\pi}{2}$ c) $tg\frac{3\pi}{2}$ d) $ctg\,\pi$ e) $\sin 2\pi$ f) $ctg\frac{\pi}{2}$ g) $\cos\frac{3\pi}{2}$ h) $tg\,\pi$ i) $\cos\frac{\pi}{2}$ j)ˇ $\sin\pi$ k)ˇ $tg\,0^\circ$ l)ˇ $ctg\frac{3\pi}{2}$ m)ˇ $\sin\frac{3\pi}{2}$ n)ˇ $ctg\,2\pi$ o)ˇ $\cos\pi$ p)ˇ $tg\frac{\pi}{2}$



Rješenje:

a) $\cos 0^\circ = 1$ b) $\sin\frac{\pi}{2} = 1$ c) $tg\frac{3\pi}{2} = -\infty$ d) $ctg\,\pi = -\infty$

e) $\sin 2\pi = 0$ f) $ctg\frac{\pi}{2} = 0$ g) $\cos\frac{3\pi}{2} = 0$ h) $tg\,\pi = 0$

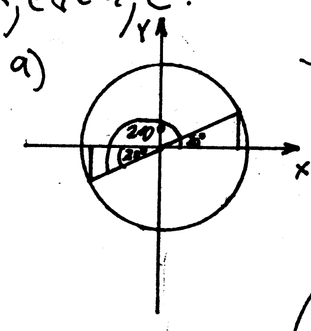
i) $\cos\frac{\pi}{2} = 0$

2. Izračunati:

a) $\sin 210^\circ$ b) $\cos 120^\circ$ c) $\sin 330^\circ$ d) $\cos 240^\circ$ e) $\sin 150^\circ$ f)ˇ $\cos 300^\circ$ g)ˇ $\sin 240^\circ$ h)ˇ $\cos 330^\circ$ i)ˇ $\sin 300^\circ$ k)ˇ $\cos 150^\circ$ l)ˇ $\sin 120^\circ$ m)ˇ $\cos 210^\circ$

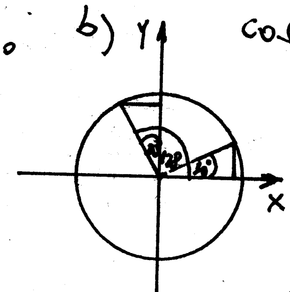
Rješenje:

a)



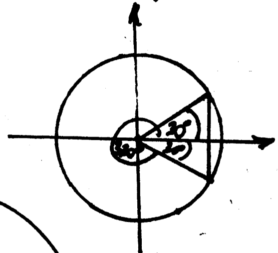
$\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$

b)



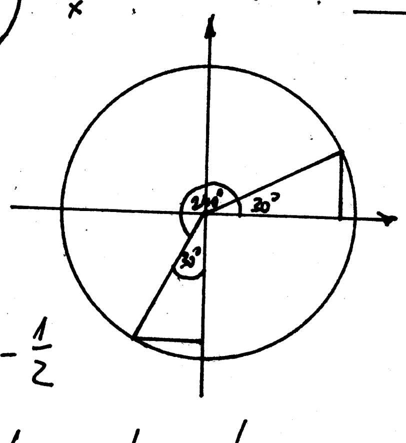
$\cos 120^\circ = -\sin 30^\circ = -\frac{1}{2}$

c)



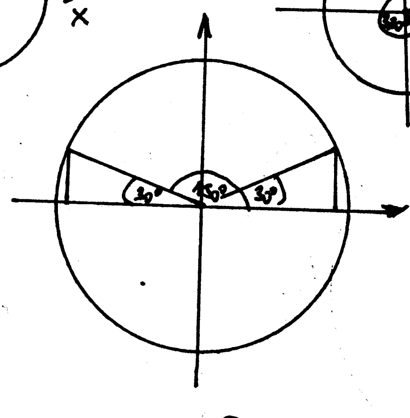
$\sin 330^\circ = -\sin 30^\circ = -\frac{1}{2}$

d)



$\cos 240^\circ = -\sin 30^\circ = -\frac{1}{2}$

e)



$\sin 150^\circ = \sin 30^\circ = \frac{1}{2}$.

3.ˇ Pojednostaviti zadane izraze:

a) $\frac{1}{\sin^2\alpha} - 1$ b) $\frac{1-\cos^2\alpha}{\sin\alpha\cos\alpha}$ c) $\frac{1+\cos^2\alpha - \sin^2\alpha}{\sin^2\alpha}$ d) $\frac{1+\sin\alpha - \cos^2\alpha}{1+\sin\alpha}$